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Hall Families and the Marriage Problem

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If the axiom of choice is accepted, every family of nonempty sets has a choice function. On the other hand there are even finite families of nonempty sets which do not possess an injective choice function. Therefore one is interested in establishing necessary and sufficient criterions which decide if a family has an injective choice function. Such a criterion is not known yet. But there are criterions which work for certain classes of families. We will give a short historical survey.

First P. Hall [5] formulated a criterion for finite families. This criterion was generalized by M. Hall [4] to arbitrary families which have finite members only. Rado and Jung [13] proved a criterion which works for families with exactly one infinite member. Brualdi and Scrimger [1], Folkman [3], McCarthy [7], Steffens [15] and Woodall [17] established criterions for families with finitely many infinite members. That the criterion in [15] also applies to countable families was shown by Podewski and Steffens in [11]. In [8] Nash–Williams formulated a criterion and conjectured that it would work for countable families. This conjecture was proved by Damerell and Milner in [2]. In [9] the result of [2] was modified and in [10] Nash–Williams described an alternative line of approach to margin functions. Shelah proved in [14] a criterion that applies for example to families F with $|F| = \aleph_1$ and countable members only. In [12] some structural results were proved. Finally [16] contains a summary of the results in [11, 12, 15] and some generalization to matroids.

In this paper it will be found that many results of the papers cited above can be generalized using Hall families, as can for example the criterion of Damerell, Milner and Nash–Williams. Hall families roughly speaking behave as families with finite members only. Therefore it is not astonishing that this concept is very powerful in its applications. In the second part of the paper we use our methods to prove a straightforward generalization of the criterion of Damerell, Milner and Nash–Williams. Nash–Williams' criterion is based on the concept of margin function. Our proof shows the connection between this concept and the concept of critical subfamilies.

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Among other things we prove in this paper the following results:

(1) A criterion deciding whether Hall families have injective choice functions.

(2) A characterization of critical families.

(3) If F is a critical family, then the set of nonempty critical subfamilies of F has a minimal element (Corollary 26).

(4) Every Hall family has a maximal representable subfamily.

(5) A generalization of the criterion of Damerell, Milner and Nash–Williams.

1. Definitions

A *family* $F = (F(i) \mid i \in I)$ is a function from an index set I into a set. A subset G of F is called a *subfamily* of F. If F is a family, $\operatorname{dmn} F$ denotes the domain of F and $\operatorname{rng} F$ denotes the range of F; if $J \subseteq \operatorname{dmn} F$, let $F \restriction J$ be the restriction of F to J and put $F(J) = \bigcup \operatorname{rng} F \restriction J$. A function f from $\operatorname{dmn} F$ into $F(\operatorname{dmn} F)$ is called a *choice function* of F if $f(i) \in F(i)$ for every $i \in \operatorname{dmn} F$. Let $IA(F)$ be the set of all injective choice functions of F. A family G is called *critical* if $IA(G) \neq \varnothing$ and $\operatorname{rng} f = G(\operatorname{dmn} G)$ for every $f \in IA(G)$. If F is a family, a subfamily M of F is called *maximal representable* if $IA(M) \neq \varnothing$ and $IA(H) = \varnothing$ for every subfamily H with $M \subsetneqq H \subseteq F$. If A is a set, put $F \backslash A = (F(i) \backslash A \mid i \in \operatorname{dmn} F)$, $F \cup A = (F(i) \cup A \mid i \in \operatorname{dmn} F)$ and $F \cap A = (F(i) \cap A \mid i \in \operatorname{dmn} F)$. $A \Subset B$ means that A is a finite subset of B.

Let $F = (F(i) \mid i \in I)$ and $S = (S(i) \mid i \in I)$ be families. We say that S has the *lifting property* (w.r.t. F) if for every $f \in IA(F)$ there is a $g \in IA(S)$ with $\operatorname{rng} g \subseteq \operatorname{rng} f$. If $F = (F(i) \mid i \in I)$, let I_H be the set of all $i \in I$ with the property that there is a finite set A and a set $J \subseteq I$ such that $F(i) \subseteq F(J) \cup A$ and $(F \restriction J) \backslash A$ is critical. $F \restriction I_H$ is called the *Hall part* of F. F is called a *Hall family* if $I = I_H$. For $F = (F(i) \mid i \in I)$ we define by transfinite recursion a sequence $(J(\alpha, F) \mid \alpha \leqslant \gamma)$ of subsets of I. Put $J(0, F) = \varnothing$. If $J(\beta, F)$ is already defined for every $\beta < \alpha$, we set $J(\alpha, F) = \bigcup \{J(\beta, F) \mid \beta < \alpha\}$ if α is a limit ordinal; if $\alpha = \beta + 1$, $i \in J(\alpha, F)$ iff there are a finite set A and a set $J \subseteq J(\beta, F)$ such that $F(i) \subseteq F(J) \cup A$ and $(F \restriction J) \backslash A$ is critical. Let γ be the least ordinal α with $J(\alpha + 1, F) \backslash J(\alpha, F) = \varnothing$. We have $J(\beta, F) \subsetneqq J(\alpha, F)$ for $\beta < \alpha \leqslant \gamma$. If the context is clear we write J_α instead of $J(\alpha, F)$. If $i \in \bigcup \{J(\alpha, F) \mid \alpha \leqslant \gamma\}$, the *rank* of i, denoted by $\operatorname{rk}(i, F)$, is the least ordinal α such that $i \in J(\alpha + 1, F)$. If the context is clear we write $\operatorname{rk}(i)$ instead of $\operatorname{rk}(i, F)$. We say that *i has a rank* if there is an ordinal $\alpha \leqslant \gamma$ with $i \in J(\alpha, F)$. The ordinal number $\operatorname{rk}(F) = \sup\{\operatorname{rk}(i, F) + 1 \mid i \in \operatorname{dmn} F \text{ and } i \text{ has a rank}\} = \gamma$ is called the *rank of F*. Note that $\operatorname{rk}(i) = 0$ iff $F(i)$ is finite.

If $F = (F(i) \mid i \in I)$ is a family such that every $i \in I$ has a rank and if $S = (S(i) \mid i \in I)$ is a family with $S(i) \Subset F(i)$ for every $i \in I$, then S is called a *finite characterization* of F if for every $i \in I$ there is a set $J \subseteq J_{\mathrm{rk}(i)}$ such that $F(i) \subseteq F(J) \cup S(i)$ and $(F \upharpoonright J)\backslash S(i)$ is critical. A set A is *jilted* w.r.t. F if there is a function $f \in IA(F)$ such that $A \cap \operatorname{rng} f = \varnothing$. If $J \subseteq I$, $K \subseteq I$, $f \in IA(F \upharpoonright J)$, $g \in IA(F \upharpoonright K)$ and $x \in F(J)$, we define a sequence $(i_n \mid n < k \leqslant \omega)$, called an *$(f, g)$-zigzag with beginning point x*, as follows: If $x \notin \operatorname{rng} f$, i_0 is undefined and $k = 0$. Otherwise put $i_0 = f^{-1}(x)$. If $i_n \in K$ is already defined and if $g(i_n) \in \operatorname{rng} f$, we set $i_{n+1} = f^{-1}(g(i_n))$; if i_n is defined and $i_n \notin K$ or $g(i_n) \notin \operatorname{rng} f$, i_{n+1} is undefined and $k = n + 1$. If i_n is defined for every $n \in \omega$, put $k = \omega$.

2. Some Well-Known Facts and Preliminary Lemmas

In this section we cite some well-known facts from [11, 12, 15, 16]. In addition we will prove some results which are frequently used and which can be found implicitly in the papers of Brualdi and Scrimger [1], Kaluza [6] and Ziegler [18].

Theorem 1 [4]. *If $|F(i)| < \aleph_0$ for every $i \in \operatorname{dmn} F$, then F has an injective choice function iff every finite subfamily of F has an injective choice function.*

Theorem 2 [11]. *If $IA(F) \neq \varnothing$ and if $X \subseteq \bigcap \{\operatorname{rng} f \mid f \in IA(F)\}$, then there is a set $J \subseteq \operatorname{dmn} F$ such that $F \upharpoonright J$ is critical and $X \subseteq F(J)$.*

Theorem 3 [15]. *Let $F = (F(i) \mid i \in I)$ be a family. If $K \subseteq I$ and $i_0 \in I\backslash K$ and $\varnothing \neq F(i_0) \subseteq F(K)$ and $F \upharpoonright K$ is critical, then there exists a $k_0 \in K$ such that $F \upharpoonright (K\backslash\{k_0\}) \cup \{i_0\}$ is critical.*

Lemma 4. *If $IA(F) \neq \varnothing$ and if G is a union of critical subfamilies of F, then G is critical.*

Theorem 5 [15]. *Let F be a critical family with $|F(i)| < \aleph_0$ for every $i \in \operatorname{dmn} F$. Then*

$$F = \bigcup \{G \subseteq F \mid G \text{ critical and } |G| < \aleph_0\}.$$

Lemma 6. *Let $F = (F(i) \mid i \in I)$ be a family with $IA(F) \neq \varnothing$. Let $K, J \subseteq I$ and $Y \subseteq F(K)$ and let $F \upharpoonright K$ and $(F \upharpoonright J)\backslash Y$ be critical. Then $F \upharpoonright J \cup K$ is critical.*

Proof. Since $IA(F) \pm \varnothing$, we have $IA(F \upharpoonright J \cup K) \neq \varnothing$. Choose $g \in IA(F \upharpoonright J \cup K)$ and assume that there is an $x \in F(J \cup K)\backslash \operatorname{rng} g$. $F \upharpoonright K$ is critical, therefore $x \notin F(K)$, in particular $x \notin Y$. Choose $h \in IA((F \upharpoonright J)\backslash Y)$; $x \in \operatorname{rng} h$ since $(F \upharpoonright J)\backslash Y$ is critical. Let $(i_n \mid n < k \leqslant \omega)$ be an $(h, g \upharpoonright J)$-zigzag with beginning point x. k must be finite, for $(F \upharpoonright J)\backslash Y$ is critical. Clearly we have $g(i_{k-1}) \in F(J)$ and $\operatorname{rng} h = F(J)\backslash Y$. Consequently $g(i_{k-1}) \in Y$. $Y \subseteq F(K) = \operatorname{rng} g \upharpoonright K$ implies $i_{k-1} \in K$. Let r be the greatest natural number j such that $i_j \notin K$. Then $g(i_r) = h(i_{r+1}) \in F(K)\backslash\operatorname{rng}(g \upharpoonright K)$ and $g \upharpoonright K \in IA(F \upharpoonright K)$ which contradicts the assumption that $F \upharpoonright K$ is critical.

COROLLARY 7. *Let A be a maximal jilted subset of Y w.r.t. $F \upharpoonright J_\alpha$. If J is a subset of J_α such that $(F \upharpoonright J)\backslash Y$ is critical, there is a set J' with the following properties*:

(1) $J \subseteq J' \subseteq J_\alpha$,

(2) $(F \upharpoonright J')\backslash A$ *is critical,*

(3) $Y \subseteq F(J') \cup A$.

Proof. A is a maximal jilted subset of Y w.r.t. $F \upharpoonright J_\alpha$, therefore $Y\backslash A \subseteq \bigcap \{\operatorname{rng} f \mid f \in IA((F \upharpoonright J_\alpha)\backslash A)\}$. By Theorem 2 there is a set $K \subseteq J_\alpha$ such that $Y\backslash A \subseteq F(K)\backslash A$ and $(F \upharpoonright K)\backslash A$ is critical. Lemma 6 now yields the corollary, if F, I, Y, $J \cup K$ are replaced by $F\backslash A$, J_α, $Y\backslash A$, J' respectively.

LEMMA 8. *Let $F = (F(i) \mid i \in I)$ be a family, let A, Y be sets and $J \subseteq I$ such that $Y \subseteq F(J) \cup A$ and $(F \upharpoonright J)\backslash A$ is critical. Then for every $f \in IA(F)$ and for every $B \subseteq Y\backslash(A \cup \operatorname{rng} f)$ there exists an injective function φ from B into $A \cap \operatorname{rng} f$ and a $g \in IA(F)$ such that $\operatorname{rng} g = (\operatorname{rng} f\backslash\varphi[B]) \cup B$.*

Proof. Choose $h \in IA((F \upharpoonright J)\backslash A)$. For every $b \in B$ let $(i_k^b \mid k < n_b \leqslant \omega)$ be an (h, f)-zigzag with beginning point b. We prove that every n_b is finite. Assume that there is a $b \in B$ such that $n_b = \omega$ and put $h' = h \upharpoonright (J\backslash\{i_k^b \mid k < \omega)\} \cup f \upharpoonright \{i_k^b \mid k < \omega\}$. We have $h' \in IA((F \upharpoonright J)\backslash A)$ and $b \in (F(J)\backslash A)\backslash\operatorname{rng} h'$, which is contradictory to the assumption that $(F \upharpoonright J)\backslash A$ is critical.

φ is defined by $\varphi(b) = f(i^b_{n_b-1})$ for $b \in B$. Then φ is injective, since h and f are injective. Clearly $\varphi(b) \in \operatorname{rng} f \cap F(J)$. $(F \upharpoonright J)\backslash A$ is critical, therefore $\varphi(b) \in A$. We now get our $g \in IA(F)$:

$$g = f \upharpoonright (I\backslash\{i_k^b \mid b \in B, k < n_b\}) \cup h \upharpoonright \{i_k^b \mid b \in B, k < n_b\}.$$

COROLLARY 9. *If A is a maximal jilted subset of Y* w.r.t. *F, then $|C| \leqslant |A|$ for every jilted subset C of Y* w.r.t. *F.*

Proof. Let A be a maximal jilted subset of Y w.r.t. F and let C be a jilted subset of Y w.r.t. F. By Theorem 2 there is a set $J \subseteq \operatorname{dmn} F$ such that $Y \subseteq F(J) \cup A$ and $(F \upharpoonright J)\backslash A$ is critical. Put $B = C\backslash A$. By assumption there is an $f \in IA(F)$ such that $C \cap \operatorname{rng} f = \varnothing$. Clearly $B \subseteq Y\backslash(A \cup \operatorname{rng} f)$. Lemma 8 yields an injective function from B into $A \cap \operatorname{rng} f$. This implies $|C| \leqslant |A|$.

Remark. Corollary 9 implies that two maximal jilted subsets have the same cardinality.

LEMMA 10. *Let F be a critical family with* $\operatorname{dmn} F = I$ *and let* $A \subseteq F(I)$. *Then there is a set* $J \subseteq I$ *such that* $F(J) \cup A = F(I)$ *and* $(F \restriction J)\backslash A$ *is critical.*

Proof. Choose $f \in IA(F)$ and $J = \{i \in I \mid f(i) \notin A\}$. J has the desired properties.

3. HALL FAMILIES

In this section we want to prove that F is a Hall family iff every $i \in \operatorname{dmn} F$ has a rank.

LEMMA 11. *Let F be a family. Then*

(i) *If A is a finite set and* $i \in \operatorname{dmn} F$ *and i has a rank* w.r.t. F, *then i has a rank* w.r.t. $F\backslash A$ *and* w.r.t. $F \cup A$ *and* $\operatorname{rk}(i, F) = \operatorname{rk}(i, F\backslash A) = \operatorname{rk}(i, F \cup A)$.

(ii) *If G is a subfamily of F,* $i \in \operatorname{dmn} G$ *and if* $\operatorname{rk}(i, G)$ *exists, then* $\operatorname{rk}(i, F)$ *exists and* $\operatorname{rk}(i, F) \leqslant \operatorname{rk}(i, G)$.

Proof. We prove by transfinite induction on α that $J(\alpha, F) = J(\alpha, F\backslash A)$. Let $G = F\backslash A$. Clearly $J(0, F) = J(0, G)$. If α is a limit ordinal then the claim follows easily from the inductive hypothesis. Now let $\alpha = \beta + 1$. If $i \in J(\alpha, G)$ then there are sets J, B such that B is finite, $J \subseteq J(\beta, G)$, $(G \restriction J)\backslash B$ is critical and $F(i) \subseteq G(J) \cup B$. This implies $J \subseteq J(\beta, F)$, $(F \restriction J)\backslash(A \cup B)$ is critical and $F(i) \subseteq F(J) \cup (A \cup B)$. Therefore $i \in J(\alpha, F)$. To prove the conserve let $i \in J(\alpha, F)$ and J, B be sets such that B is finite, $J \subseteq J(\beta, F) = J(\beta, G)$, $(F \restriction J)\backslash B$ is critical and $F(i) \subseteq F(J) \cup B$. We apply Lemma 10 and replace F, I, J, A by $(F \restriction J)\backslash B$, J, J', $(A \cap F(J))\backslash B$. Therefore $J' \subseteq J \subseteq J(\beta, G)$, $((F \restriction J')\backslash B)\backslash((A \cap F(J))\backslash B) = ((F \restriction J')\backslash A)\backslash B = (G \restriction J')\backslash B$ is critical and $(F(J')\backslash B) \cup ((A \cap F(J))\backslash B) = F(J)\backslash B$. Therefore $(F(J)\backslash A) \cup B \subseteq (F(J')\backslash A) \cup B$ and consequently $G(i) = F(i)\backslash A \subseteq (F(J)\backslash A) \cup B \subseteq (F(J')\backslash A) \cup B = G(J') \cup B$. This implies that $i \in J(\alpha, G)$.

The corresponding proof for the family $F \cup A$ and the proof of (ii) are left to the reader.

LEMMA 12. *If F is a critical family, then every* $i \in \operatorname{dmn} F$ *has a rank.*

Proof. Let F be a critical family with $I = \operatorname{dmn} F$, choose $f \in IA(F)$ and assume that there is an ordinal α such that $J_{\alpha+1}\backslash J_\alpha = \varnothing$ and $I\backslash J_\alpha \neq \varnothing$.

Choose $i_0{}^0 \in I\backslash J_\alpha$, put $A_0 = \{f(i_0{}^0)\}$ and $(i_0{}^0, \ldots, i_{n_0}^0) = (i_0{}^0)$. For $k < \omega$

define $J_\alpha{}^k \subset J_\alpha$, $h_k \in IA(F \upharpoonright J_\alpha{}^k)$, (f, h_k)-zigzags $(i_0{}^k, \ldots, i^k_{n_k})$ $(k \neq 0)$ and sets $A_k = \{f(i_j{}^k) \mid 0 \leqslant j \leqslant n_k\}$ with the following properties:

$$\begin{aligned}
&(1)\quad f(i_0^{k+1}) \in F(i^k_{n_k}) \backslash \operatorname{rng} h_{k+1}\,,\\
&(2)\quad \bigcup\{A_j \mid j < k\} \cap \operatorname{rng} h_k = \varnothing,\\
&(3)\quad i^k_{n_k} \in I \backslash J_\alpha\,.
\end{aligned}$$

For $k = 0$ define $h_0 = f$ and $J_\alpha{}^0 = J_\alpha$. Assume that for $k = m$ everything is defined. Put $J_\alpha^{m+1} = J_\alpha \backslash \{i_k{}^l \mid l \leqslant m, k \leqslant n_l\}$.

CLAIM. There are $a \in F(i^m_{n_m}) \backslash \bigcup \{A_j \mid j \leqslant m\}$ and $h \in IA((F \upharpoonright J_\alpha^{m+1}) \backslash \bigcup \{A_j \mid j \leqslant m\})$ such that $a \notin \operatorname{rng} h$.

Assume the contrary. Then for every $h \in IA((F \upharpoonright J_\alpha^{m+1}) \backslash \bigcup \{A_j \mid j \leqslant m\})$ we get $F(i^m_{n_m}) \backslash \bigcup \{A_j \mid j \leqslant m\} \subseteq \operatorname{rng} h$. With (2) one can easily show that

$$h_m \upharpoonright J_\alpha^{m+1} \in IA\left((F \upharpoonright J_\alpha^{m+1}) \backslash \bigcup \{A_j \mid j \leqslant m\}\right).$$

Therefore Theorem 2 yields a set $J \subseteq J_\alpha^{m+1}$ such that $(F \upharpoonright J) \backslash \bigcup \{A_j \mid j \leqslant m\}$ is critical and $F(i^m_{n_m}) \subseteq F(J) \cup \bigcup \{A_j \mid j \leqslant m\}$. $\bigcup \{A_j \mid j \leqslant m\}$ is finite, hence $i^m_{n_m} \in J_{\alpha+1} \backslash J_\alpha$; this is a contradiction.

Choose a and $h = h_{m+1}$ according to the claim and define $i_0^{m+1} = f^{-1}(a)$. If $i_0^{m+1} \notin J_\alpha$, then $n_{m+1} := 0$. Let $i_0^{m+1}, \ldots, i_k^{m+1}$ be already defined. If $i_k^{m+1} \in J_\alpha^{m+1}$, put $i_{k+1}^{m+1} = f^{-1}(h_{m+1}(i_k^{m+1}))$. F is critical, hence there is an n_{m+1} such that $i^{m+1}_{n_{m+1}} \notin J_\alpha$. Now we define the function f' by

$$f' = (f \backslash \{(i_k{}^m, f(i_k{}^m)) \mid m \in \omega, k \leqslant n_m\}) \cup \{(i_k{}^m, h_m(i_k{}^m)) \mid m \in \omega, k < n_m\}$$
$$\cup \{(i^m_{n_m}, f(i_0^{m+1})) \mid m \in \omega\}.$$

We have $f' \in IA(F)$ and $f(i_0{}^0) \notin \operatorname{rng} f'$, contradicting the fact that F is critical. Thus our supposition that there exists an ordinal α with $J_{\alpha+1} \backslash J_\alpha = \varnothing$ and $I \backslash J_\alpha = \varnothing$ has led to a contradiction, and so Lemma 12 is proved.

THEOREM 13. *F is a Hall family iff every $i \in \operatorname{dmn} F$ has a rank.*

Proof. Let F be a Hall family and suppose that there is an $i \in \operatorname{dmn} F$ without rank. Then $F(i)$ is infinite. By definition there are sets A and J such that A is finite, $J \subseteq \operatorname{dmn} F$, $F(i) \subseteq F(J) \cup A$ and $(F \upharpoonright J) \backslash A$ is critical. We can apply Theorem 3 to the family $F \backslash A$, since $F(i) \backslash A \neq \varnothing$ and $F(i) \backslash A \subseteq F(J) \backslash A$, and get a set $K \subseteq \operatorname{dmn} F$ such that $i \in K$ and $(F \upharpoonright K) \backslash A$ is critical. By Lemma 12, i has a rank in $(F \upharpoonright K) \backslash A$, and with aid of Lemma 11 we can show that i has a rank in F. This is a contradiction. To prove the converse we assume that every $i \in \operatorname{dmn} F$ has a rank. Let $i \in \operatorname{dmn} F$ and let $\alpha = \operatorname{rk}(i)$. Then there exist a finite set A and a set $J \subseteq J_\alpha$ such that $(F \upharpoonright J)\backslash A$ is critical and $F(i) \subseteq F(J) \cup A$. Consequently we get $i \in I_H$.

With similar methods one can show by transfinite induction:

LEMMA 14. *If F is a family with $I = \operatorname{dmn} F$ and if $F \upharpoonright I_H$ is the Hall part of F, then $J(\alpha, F) = J(\alpha, F \upharpoonright I_H)$ for every ordinal α.*

4. FINITE CHARACTERIZATIONS

First we want to prove that every Hall family has a finite characterization. We need the following lemma.

LEMMA 15. *Let F be a family and J be a set with $J \subseteq \operatorname{dmn} F$, let Y be a set and A be a finite set such that $(F \upharpoonright J)\backslash A$ is critical and $Y \subseteq F(J) \cup A$. Then there exist sets $J' \subseteq J$ and A' such that $A \cap Y \subseteq A' \subseteq Y$, $|A'| \leqslant |A|$, $Y \subseteq F(J') \cup A'$ and $(F \upharpoonright J')\backslash A'$ is critical.*

Proof. By assumption A is a maximal jilted subset of $Y \cup A$ w.r.t. $F \upharpoonright J$. Let B be a jilted subset of $Y \cup A$ w.r.t. $F \upharpoonright J$ such that $Y \cap A \subseteq B \subseteq Y$. Then, by Corollary 9, we have $|B| \leqslant |A|$. Hence there is a maximal set A' such that $|A'| \leqslant |A|$, $A \cap Y \subseteq A' \subseteq Y$ and $IA((F \upharpoonright J)\backslash A') \neq \varnothing$. Theorem 2 yields a set $J' \subseteq J$ with the property that $Y \subseteq F(J') \cup A'$ and $(F \upharpoonright J')\backslash A'$ is critical.

THEOREM 16. *Every Hall family has a finite characterization.*

Proof. Let F be a Hall family. Then, by Theorem 13, every $i \in \operatorname{dmn} F$ has a rank. If $\operatorname{rk}(i) = \alpha$, then there are a finite set A and a set $J \subseteq J_\alpha$ such that $(F \upharpoonright J)\backslash A$ is critical and $F(i) \subseteq F(J) \cup A$. Put $Y = F(i)$. Then, by Lemma 15, there are sets A' and J' such that $A' \Subset F(i)$, $J' \subseteq J$, $F(i) \subseteq F(J') \cup A'$ and $(F \upharpoonright J')\backslash A'$ is critical. Put $S(i) = A'$ for $i \in \operatorname{dmn} F$. The family $(S(i) \mid i \in \operatorname{dmn} F)$ is a finite characterization of F.

In the following passage we prove a necessary and sufficient criterion for the existence of an injective choice function of Hall families.

LEMMA 17. *Let $F = (F(i) \mid i \in I)$ and $S = (S(i) \mid i \in I)$ be families such that $S(i) \Subset F(i)$ for every $i \in I$. If for every $K \Subset I$ there is a set $J \supseteq K$ such that $IA(F \upharpoonright J) \neq \varnothing$ and $S \upharpoonright J$ has the lifting property* w.r.t. *$F \upharpoonright J$, then $IA(F) \neq \varnothing$ and S has the lifting property* w.r.t. *F.*

Proof. By Theorem 1 we have $IA(S) \neq \varnothing$. Choose $h \in IA(F)$. Obviously it suffices to show: $IA(S \cap \operatorname{rng} h) \neq \varnothing$.

Suppose that $IA(S \cap \operatorname{rng} h) = \varnothing$. Then, by Theorem 1, there is a set

$K \Subset I$ with $IA(S \upharpoonright K \cap \operatorname{rng} h) = \varnothing$. By assumption there is a $J \supseteq K$ such that $S \upharpoonright J$ has the lifting property w.r.t. $F \upharpoonright J$. Hence there exists a $g \in IA(S \upharpoonright J)$ with $\operatorname{rng} g \subseteq \operatorname{rng} h \upharpoonright J \subseteq \operatorname{rng} h$. This implies $g \upharpoonright K \in IA(S \upharpoonright K \cap \operatorname{rng} h)$. Thus our supposition has led to a contradiction, and so Lemma 17 is proved.

LEMMA 18. *Let $F = (F(i) \mid i \in I)$ be a Hall family and let $S = (S(i) \mid i \in I)$ be a finite characterization of F. Let J be a set such that $J_\alpha \subseteq J \subseteq J_{\alpha+1}$, $IA(F \upharpoonright J) \neq \varnothing$ and $S \upharpoonright J$ has the lifting property* w.r.t. *$F \upharpoonright J$. If $i \in J_{\alpha+1}$ and $IA(F \upharpoonright J \cup \{i\}) \neq \varnothing$, then $S \upharpoonright J \cup \{i\}$ has the lifting property* w.r.t. *$F \upharpoonright J \cup \{i\}$.*

Proof. If $i \in J$, we have nothing to prove. Hence. assume $i \in J_{\alpha+1} \backslash J$ and choose $g \in IA(F \upharpoonright J \cup \{i\})$.

CLAIM. There exists an $h \in IA(F \upharpoonright J \cup \{i\})$ such that $h(i) \in S(i)$ and $\operatorname{rng} g = \operatorname{rng} h$.

If $g(i) \in S(i)$, put $h = g$. Now assume $g(i) \notin S(i)$. Then, by definition of $S(i)$, there is a set $J' \subseteq J_\alpha \subseteq J$ such that $(F \upharpoonright J') \backslash S(i)$ is critical and $F(i) \subseteq F(J') \cup S(i)$. Set $Y = F(i)$, $A = S(i)$, $f = g \upharpoonright J$ and $B = \{g(i)\}$. By Lemma 8, where the objects F, I, J, φ, g of Lemma 8 are here replaced by $F \upharpoonright J, J, J', \{(g(i), a)\}$, h' respectively, there are a function $h' \in IA(F \upharpoonright J)$ and an $a \in S(i) \cap \operatorname{rng} g \upharpoonright J$ such that $\operatorname{rng} h' = ((\operatorname{rng} g \upharpoonright J) \backslash \{a\}) \cup \{g(i)\}$. This implies $h = h' \cup \{(i, a)\} \in IA(F \upharpoonright J \cup \{i\})$ and $\operatorname{rng} h = \operatorname{rng} g$, which proves the claim.

Since $S \upharpoonright J$ has the lifting property w.r.t. $F \upharpoonright J$, there is an $f' \in IA(S \upharpoonright J)$ with $\operatorname{rng} f' \subseteq \operatorname{rng} h \upharpoonright J$. For $f = f' \cup \{(i, h(i))\}$ we get $f \in IA(S \upharpoonright J \cup \{i\})$ and $\operatorname{rng} f \subseteq \operatorname{rng} g$; and Lemma 18 is proved.

LEMMA 19. *Let $F = (F(i) \mid i \in I)$ be a family and let $S = (S(i) \mid i \in I)$ be a finite characterization of F. If $S \upharpoonright J_\alpha$ has the lifting property* w.r.t. *$F \upharpoonright J_\alpha$ and if $K \Subset J_{\alpha+1}$ is a set with $IA(F \upharpoonright (J_\alpha \cup K)) \neq \varnothing$, then $S \upharpoonright (J_\alpha \cup K)$ has the lifting property* w.r.t. *$F \upharpoonright (J_\alpha \cup K)$.*

Proof. The lemma can be proved by induction on $|K|$. Lemma 18 yields the induction step.

THEOREM 20. *Let $F = (F(i) \mid i \in I)$ be a Hall family. Then F has an injective choice function iff for every ordinal α with $IA(F \upharpoonright J_\alpha) \neq \varnothing$ and for every set $K \Subset J_{\alpha+1}$ we have $IA(F \upharpoonright (J_\alpha \cup K)) \neq \varnothing$.*

Proof. Obviously the criterion is necessary. Now assume that F is a Hall family which satisfies the conditions in Theorem 20. Then, by Theorem 16, F has a finite characterization S. We prove by transfinite induction on α that $IA(F \upharpoonright J_\alpha) \neq \varnothing$ and that $S \upharpoonright J_\alpha$ has the lifting property w.r.t. $F \upharpoonright J_\alpha$. This is obvious for $\alpha = 0$. Assume that $IA(F \upharpoonright J_\beta) \neq \varnothing$ and that $S \upharpoonright J_\beta$ has the lifting property w.r.t. $F \upharpoonright J_\beta$ for every $\beta < \alpha$. If α is a limit ordinal, Lemma 17 yields the induction step. If $\alpha = \beta + 1$, then we conclude by assumption and from Lemma 19 that $S \upharpoonright (J_\beta \cup K)$ has the lifting property w.r.t. $F \upharpoonright (J_\beta \cup K)$ for every $K \Subset J_\alpha$. Again by Lemma 17 we obtain: $IA(F \upharpoonright J_\alpha) \neq \varnothing$ and $S \upharpoonright J_\alpha$ has the lifting property w.r.t. $F \upharpoonright J_\alpha$. Since by Theorem 13 we have $I = J_\gamma$ for an ordinal γ Theorem 20 is proved.

The proof of Theorem 20 yields immediately:

THEOREM 21. *If F is a Hall family and if S is a finite characterization of F, then S has the lifting property* w.r.t. F.

THEOREM 22. *If $F = (F(i) \mid i \in I)$ is a family such that $I \backslash I_H$ is countable, then $IA(F) \neq \varnothing$ iff $IA(F \upharpoonright I_H) \neq \varnothing$.*

Proof. If $IA(F) \neq \varnothing$, then clearly $IA(F \upharpoonright I_H) \neq \varnothing$. Assume that $IA(F \upharpoonright I_H) \neq \varnothing$. Let $(i_n \mid n \in \omega)$ be an enumeration of $I \backslash I_H$ and let S_H be a finite characterization of $F \upharpoonright I_H$. Then, by Theorem 21, S_H has the lifting property w.r.t. $F \upharpoonright I_H$. We define by induction a sequence $(f_n \mid n \in \omega)$ of partial injective choice functions. For $k < n < \omega$ let f_k be an injective choice function of $F \upharpoonright \{i_r \mid r < k\}$ with the following properties:

$$(1)\quad IA(S_H \backslash \operatorname{rng} f_k) \neq \varnothing$$
$$(2)\quad f_{k_1} \subseteq f_{k_2} \qquad \text{for} \quad k_1 < k_2 < n.$$

First suppose that $F(i_{n-1}) \backslash \operatorname{rng} f_{n-1} \subseteq \operatorname{rng} f$ for every $f \in IA((F \upharpoonright I_H) \backslash \operatorname{rng} f_{n-1})$. Then, by Theorem 2, we get a set $J \subseteq I_H$ such that $(F \upharpoonright J) \backslash \operatorname{rng} f_{n-1}$ is critical and $F(i_{n-1}) \backslash \operatorname{rng} f_{n-1} \subseteq F(J) \backslash \operatorname{rng} f_{n-1}$. Hence $i_{n-1} \in I_H$, which contradicts $i_{n-1} \in I \backslash I_H$. Consequently there is a function $f \in IA((F \upharpoonright I_H) \backslash \operatorname{rng} f_{n-1})$ such that $F(i_{n-1}) \backslash (\operatorname{rng} f \cup \operatorname{rng} f_{n-1}) \neq \varnothing$. Choose $x \in F(i_{n-1}) \backslash (\operatorname{rng} f \cup \operatorname{rng} f_{n-1})$ and put $f_n = f_{n-1} \cup \{(i_{n-1}, x)\}$. S_H has the lifting property w.r.t. $F \upharpoonright I_H$, hence there exists a function $h \in IA(S_H)$ such that $\operatorname{rng} h \subseteq \operatorname{rng} f$. Clearly $x \notin \operatorname{rng} h$; consequently we get $IA(S_H \backslash \operatorname{rng} f_n) \neq \varnothing$. Now $f = \bigcup \{f_n \mid n \in \omega\}$ is an injective choice function of $F \upharpoonright (I \backslash I_H)$. With Theorem 1 we can prove the existence of a $g \in IA(S_H \backslash \operatorname{rng} f)$. Clearly we have $f \cup g \in IA(F)$; and Theorem 22 is proved.

Remark. From Theorem 20 and Theorem 22 we obtain the criterion of [10] for countable families F:

$$IA(F) \neq \varnothing \qquad \text{iff} \quad \forall K \subseteq \operatorname{dmn} F \ \forall i \in \operatorname{dmn} F \backslash K$$
$$(F(i) \subseteq F(K) \text{ implies } F \upharpoonright K \text{ is not critical}).$$

Proof. Obviously the condition is necessary. By Theorem 22 and Theorem 20 we have only to show that for every α with $IA(F \upharpoonright J_\alpha) \neq \varnothing$ and for every $L \Subset J_{\alpha+1}$ we have $IA(F \upharpoonright (J_\alpha \cup L)) \neq \varnothing$. Assume the contrary. Then there is an α and an $L \Subset J_{\alpha+1}$ such that $IA(F \restriction J_\alpha) \neq \varnothing$ and $IA(F \restriction (J_\alpha \cup L)) = \varnothing$. Choose L minimal w.r.t. this property and put $J = J_\alpha \cup (L \backslash \{i\})$ for some $i \in L$. Then $IA(F \restriction J) \neq \varnothing$ and for every $f \in IA(F \restriction J)$ we have $F(i) \subseteq \operatorname{rng} f$. Therefore by Theorem 2 there is a $K \subseteq J$ such that $F \restriction K$ is critical and $F(i) \subseteq F(K)$, which contradicts the hypothesis.

5. Structural Properties of Hall Families and of Critical Families

First we want to show that every Hall family has a maximal representable subfamily. Next we shall investigate critical families. Using a generalization of Theorem 5 we shall show for every nonempty critical family F that the set

$$\{G \subseteq F \mid G \neq \varnothing \text{ and } G \text{ critical}\}$$

has minimal elements.

Lemma 23. *Let $F = (F(i) \mid i \in I)$ be a family. Let A be a finite set, Y be a set and $J \subseteq I$ such that $Y \subseteq F(J) \cup A$ and $(F \restriction J)\backslash A$ is critical. If $K \subseteq I$ and if $F \restriction K$ is a maximal representable subfamily of F, then there exists a set $J' \subseteq K$ such that $Y \subseteq F(J') \cup A$ and $(F \restriction J')\backslash A$ is critical.*

Proof. Because $IA(F \restriction K) \neq \varnothing$ and A is finite there is a set $A' \subseteq A$ which is maximal w.r.t. the property that $IA((F \restriction K)\backslash A') \neq \varnothing$. We shall show

$$\forall h \in IA((F \restriction K)\backslash A')(Y \backslash A' \subseteq \operatorname{rng} h). \tag{*}$$

Assume the contrary. Then there are $b \in Y \backslash A'$ and $h \in IA(F \restriction K \backslash A')$ such that $b \notin \operatorname{rng} h$, since A' is maximal. Choose $f \in IA((F \restriction J)\backslash A)$ and let $(i_n \mid n < k \leqslant \omega)$ be an (f, h)-zigzag with beginning point b.

Case 1. $k = \omega$. Then $f' = (f \backslash \{(i_j, f(i_j)) \mid j < \omega\}) \cup \{(i_j, h(i_j)) \mid j < \omega\}$ is an element of $IA((F \restriction J)\backslash A)$ with $b \notin \operatorname{rng} f'$. This contradicts the assumption that $(F \restriction J)\backslash A$ is critical.

Case 2. $k < \omega$.

$$i_{k-1} \notin \operatorname{dmn} h. \tag{2.1}$$

Define $h' = (h \backslash \{(i_j, h(i_j)) \mid j < k - 1\}) \cup \{(i_j, f(i_j)) \mid j < k\}$. We have $h' \in IA(F \restriction K \cup \{(i_{k-1}, F(i_{k-1}))\})$, and this contradicts the assumption that $F \restriction K$ is maximal representable.

$$i_{k-1} \in \operatorname{dmn} h \quad \text{and} \quad h(i_{k-1}) \notin F(J)\backslash A. \tag{2.2}$$

Clearly $i_{k-1} \in J$ and $h(i_{k-1}) \in F(J)$, hence $h(i_{k-1}) \in A$. Put $h' = (h \backslash \{(i_j, h(i_j)) \mid$

$j < k\}) \cup \{(i_j, f(i_j)) \mid j < k\}$. We have $h' \in IA((F \restriction K)\backslash A')$ and $h(i_{k-1}) \in (A\backslash A')\backslash \operatorname{rng} h$; this contradicts the choice of A'.

Thus our supposition that (*) does not hold has led to a contradiction; and (*) is proved. From (*) and Theorem 2 we infer that there is a set $J'' \subseteq K$ such that $(F \restriction J'')\backslash A'$ is critical and $Y \subseteq F(J'') \cup A'$. The existence of our desired J' now follows from Lemma 10, if F, I, A, J are replaced by $(F \restriction J'')\backslash A'$, J'', $(A\backslash A') \cap F(J'')$, J' respectively. In detail: Lemma 10 implies that the family $(((F \restriction J')\backslash A')\backslash((A\backslash A') \cap F(J''))$ (this is $(F \restriction J')\backslash A)$ is critical and $F(J'')\backslash A' = (F(J')\backslash A') \cup ((A\backslash A') \cap F(J''))$. This equality yields $F(J'') \cup A' \subseteq F(J') \cup A$ and therefore $Y \subseteq F(J') \cup A$, and Lemma 23 is proved.

THEOREM 24. *Every Hall family has a maximal representable subfamily.*

Proof. Let α be the rank of F and let S be a finite characterization of F. We define by transfinite induction an increasing chain $(I_\beta \mid \beta \leqslant \alpha)$ of subsets of $\operatorname{dmn} F$ such that $S \restriction I_\beta$ is a maximal representable subfamily of $S \restriction J_\beta$.

Put $I_0 = \varnothing$. Let I_β be defined for $\beta < \rho$ with the property that $S \restriction I_\beta$ is a maximal representable subfamily of $S \restriction J_\beta$. If ρ is a limit ordinal, we set $I_\rho = \bigcup \{I_\beta \mid \beta < \rho\}$. If $\rho = \beta + 1$, then by Zorn's Lemma and Theorem 1 we get a maximal representable subfamily M of $S \restriction J_\rho$ such that $S \restriction I_\beta \subseteq M$. Define $I_\rho = \operatorname{dmn} M$.

By transfinite induction on ρ we prove the following

CLAIM. For every ordinal $\rho \leqslant \alpha$ we have

(1_ρ) $J(\gamma, F \restriction I_\rho) = I_\gamma$ for every $\gamma \leqslant \rho$.

(2_ρ) $S \restriction I_\rho$ is a finite characterization of $F \restriction I_\rho$.

(3_ρ) $F \restriction I_\rho$ is a maximal representable subfamily of $F \restriction J(\rho, F)$.

For $\rho = 0$ the proof is trivial. Now assume as inductive hypothesis that the claim is true for every ordinal $\beta < \rho$. First we prove (1_ρ) by transfinite induction on $\gamma \leqslant \rho$. For $\gamma = 0$ the proof is obvious. If γ is a limit ordinal, then by inductive hypothesis

$$J(\gamma, F \restriction I_\rho) = \bigcup \{J(\sigma, F \restriction I_\rho) \mid \sigma < \gamma\} = \bigcup \{I_\sigma \mid \sigma < \gamma\} = I_\gamma .$$

Now let $\gamma = \mu + 1$. Let $i \in J(\gamma, F \restriction I_\rho)$. Then $\operatorname{rk}(i, F \restriction I_\rho) \leqslant \mu$. By Lemma 11 we have $\operatorname{rk}(i, F) \leqslant \operatorname{rk}(i, F \restriction I_\rho) \leqslant \mu$. Therefore $i \in J(\gamma, F)$ and hence $i \in J(\gamma, F) \cap I_\rho = I_\gamma$. To prove the other inclusion let $i \in I_\gamma$. If $i \in I_\mu$, then $i \in J(\mu, F \restriction I_\rho) \subseteq J(\gamma, F \restriction I_\rho)$. Now let $i \in I_\gamma \backslash I_\mu$. Then $i \in J(\gamma, F)\backslash J(\mu, F)$ and therefore there are $J \subseteq J(\mu, F)$ and a finite set A such that $(F \restriction J)\backslash A$ is critical and $F(i) \subseteq F(J) \cup A$. By inductive hypothesis (3_μ) we have that $F \restriction I_\mu$ is a maximal representable subfamily of $F \restriction J(\mu, F)$. Lemma 23 yields a set $J' \subseteq I_\mu$ such that $F(i) \subseteq F(J') \cup A$ and $(F \restriction J')\backslash A$ is critical. By inductive hypothesis we have $I_\mu = J(\mu, F \restriction I_\rho)$. Therefore $i \in J(\gamma, F \restriction I_\rho)$, and (1_ρ) is proved.

As a consequence of (1_ρ) we prove

(4_ρ) $\quad \mathrm{rk}(i, F \restriction I_\gamma) = \mathrm{rk}(i, F \restriction I_\rho)$ for every $\gamma \leqslant \rho$ and $i \in I_\gamma$.

By Lemma 11 we have $\mathrm{rk}(i, F \restriction I_\rho) \leqslant \mathrm{rk}(i, F \restriction I_\gamma)$. To prove the other inequality put $\sigma = \mathrm{rk}(i, F \restriction I_\rho)$. Since $i \in I_\rho = J(\rho, F \restriction I_\rho)$ we get by the definition of the rank function $\sigma < \rho$. Hence $\sigma + 1 \leqslant \rho$ and therefore by (1_ρ) $J(\sigma + 1, F \restriction I_\rho) = I_{\sigma+1}$ and $i \in I_{\sigma+1}$. Because $i \notin J(\sigma, F \restriction I_\rho) = I_\sigma$ and $i \in I_\gamma$ we have $\sigma + 1 \leqslant \gamma$ and, by (1_γ), $i \in I_{\sigma+1} = J(\sigma + 1, F \restriction I_\gamma)$. Hence $\mathrm{rk}(i, F \restriction I_\gamma) \leqslant \sigma$, and (4_ρ) is rpoved.

To prove (2_ρ) and (3_ρ) we first consider the case that ρ is a limit ordinal. Let $i \in I_\rho$. Then $i \in I_\beta$ for some $\beta < \rho$. By (2_β) there is a set $J \subseteq J(\mathrm{rk}(i, F \restriction I_\beta), F \restriction I_\beta)$ such that $(F \restriction J)\backslash S(i)$ is critical and $F(i) \subseteq F(J) \cup S(i)$. Since $i \in I_\beta = J(\beta, F \restriction I_\beta)$ we have $\mathrm{rk}(i, F \restriction I_\beta) \leqslant \beta$. Thus we get with (1_β), (4_ρ) and (1_ρ):

$$\begin{aligned} J \subseteq J(rk(i, F \restriction I_\beta), F \restriction I_\beta) &= I_{rk(i, F \restriction I_\rho)} = I_{rk(i, F \restriction I_\rho)} \\ &= J(rk(i, F \restriction I_\rho), F \restriction I_\rho). \end{aligned}$$

This proves (2_ρ). The truth of (3_ρ) is easily inferred from (2_ρ) and the truth of (3_β) for all $\beta < \rho$.

To prove (2_ρ) in the case that ρ is a successor ordinal let $\rho = \beta + 1$ and $i \in I_\rho$. If $i \in I_\beta$, then the same argument as in the proof of (2_ρ) in the limit case works. Let $i \in I_\rho \backslash I_\beta$. Since $i \in J(\rho, F) \backslash J(\beta, F)$, there is a set $J \subseteq J(\beta, F)$ such that $(F \restriction J)\backslash S(i)$ is critical and $F(i) \subseteq F(J) \cup S(i)$. By (3_β) and Lemma 23 we get a set $J' \subseteq I_\beta = J(\beta, F \restriction I_\rho)$ such that $(F \restriction J')\backslash S(i)$ is critical and $F(i) \subseteq F(J) \cup S(i)$. This implies $\mathrm{rk}(i, F \restriction I_\rho) \leqslant \beta$. On the other hand we have by (1_ρ) $i \notin J(\gamma, F \restriction I_\rho)$ for all $\gamma < \rho$. Therefore $\mathrm{rk}(i, F \restriction I_\rho) \geqslant \beta$ and consequently $\mathrm{rk}(i, F \restriction I_\rho) = \beta$. Hence $J' \subseteq J(\mathrm{rk}(i, F \restriction I_\rho), F \restriction I_\rho)$, and (2_ρ) is proved.

For the proof of (3_ρ) in the case $\rho = \beta + 1$ assume the contrary and take an $i \in J(\rho, F) \backslash I_\rho$ such that there is an $f \in IA(F \restriction (I_\rho \cup \{i\}))$. By (3_β) we have $i \notin J(\beta, F) \supseteq I_\beta$. As usual Lemma 23 yields a set $J' \subseteq I_\beta$ such that $(F \restriction J')\backslash S(i)$ is critical and $F(i) \subseteq F(J') \cup S(i)$. Therefore an easy induction and (1_ρ) show that

$$J(\sigma, F \restriction I_\rho) = J(\sigma, F \restriction (I_\rho \cup \{i\})) = I_\sigma \qquad \text{for all } \sigma < \rho$$

and $\hfill (*)$

$$J(\rho, F \restriction (I_\rho \cup \{i\})) = J(\rho, F \restriction I_\rho) \cup \{i\} = I_\rho \cup \{i\}.$$

Hence every $j \in I_\rho \cup \{i\}$ has a rank in the family $F \restriction (I_\rho \cup \{i\})$. By Theorem 13 $F \restriction (I_\rho \cup \{i\})$ is a Hall family. We now show that $S \restriction (I_\rho \cup \{i\})$ is a finite characterization of $F \restriction (I_\rho \cup \{i\})$. Let $k \in I_\rho$. By $(*)$ and (2_ρ) there is a set $J \subseteq J(\mathrm{rk}(k, F \restriction (I_\rho \cup \{i\})), F \restriction (I_\rho \cup \{i\}))$ such that $(F \restriction J)\backslash S(k)$ is critical and

$F(k) \subseteq F(J) \cup S(k)$. If $k = i$, the set J' defined above has the desired properties, since by (*) $\mathrm{rk}(i, F \upharpoonright (I_\rho \cup \{i\})) = \beta$. Theorem 21 implies that $S \upharpoonright (I_\rho \cup \{i\})$ has the lifting property w.r.t. $F \upharpoonright (I_\rho \cup \{i\})$. Thus there is an $h \in IA(S \upharpoonright (I_\rho \cup \{i\}))$ with $\mathrm{rng}\, h \subseteq \mathrm{rng} f$. This contradicts the construction of I_ρ. Now we have proved (3_ρ), and this completes the proof of our claim.

By Theorem 13 and (3_α) $F \upharpoonright I_\alpha$ is a maximal representable subfamily of $F \upharpoonright J_\alpha = F$, and Theorem 24 is proved.

THEOREM 25. *If* $F = (F(i) \mid i \in I)$ *is a critical family, then* $I = \bigcup \{J \subseteq I \mid F \upharpoonright J$ *critical and* $\exists \alpha (J \subseteq J_{\alpha+1}$ *and* $1 \leqslant | J \backslash J_\alpha | < \aleph_0)\}$.

Proof. Let S be a finite characterization of F. Then, by Theorem 21, S is critical and consequently we have by Theorem 5:

$$I = \bigcup \{K \Subset I \mid S \upharpoonright K \text{ critical}\}.$$

Choose $K \Subset I$ such that $S \upharpoonright K$ is critical. It suffices to prove the existence of a set $J \subseteq I$ and of an ordinal α with $K \subseteq J$, $K \backslash J_\alpha = J \backslash J_\alpha$, $1 \leqslant | J \backslash J_\alpha |$ and $F \upharpoonright J$ critical.

K is finite, hence there is an ordinal α with $K \backslash J_\alpha \neq \varnothing$ and $K \backslash J_{\alpha+1} = \varnothing$. Set $L = K \backslash J_\alpha$, choose $g \in IA(S)$ and define $A' = \mathrm{rng}\, g \upharpoonright L$.

CLAIM 1. A' is a maximal jilted subset of $S(K)$ w.r.t. $S \upharpoonright J_\alpha$.

A' is a jilted subset of $S(K)$ w.r.t. $S \upharpoonright J_\alpha$, since $g \upharpoonright J_\alpha \in IA(S \upharpoonright J_\alpha)$. If $h \in IA(S \upharpoonright K \cap J_\alpha)$, we have $| S(K) \backslash \mathrm{rng}\, h | = | S(K) | - | \mathrm{rng}\, h | = | K | - (| K | - | L |) = | L |$. Since $g \upharpoonright K \cap J_\alpha \in IA(S \upharpoonright K \cap J_\alpha)$ and $| A' | = | L |$, we get that A' is a maximal jilted subset of $S(K)$ w.r.t. $S \upharpoonright K \cap J_\alpha$ and consequently w.r.t. $S \upharpoonright J_\alpha$.

CLAIM 2. A' is a maximal jilted subset of $S(K)$ w.r.t. $F \upharpoonright J_\alpha$.

$S \upharpoonright J_\alpha$ is a finite characterization of $F \upharpoonright J_\alpha$. Hence, by Theorem 21, $S \upharpoonright J_\alpha$ has the lifting-property w.r.t. $F \upharpoonright J_\alpha$. Consequently every jilted subset of $S(K)$ w.r.t. $F \upharpoonright J_\alpha$ is a subset of a jilted subset of $S(K)$ w.r.t. $S \upharpoonright J_\alpha$.

CLAIM 3. There exists a set $J' \subseteq J_\alpha$ such that $(F \upharpoonright J') \backslash A'$ is critical and $F(K) \subseteq F(J') \cup A'$.

By Claim 2 A' is a maximal jilted subset of $Y_i = S(i) \cup A'$ w.r.t. $F \upharpoonright J_\alpha$ for every $i \in K$. Because S is a finite characterization of F, there exists for every $i \in K$ a set $J_i^1 \subseteq J_\alpha$ such that $(F \upharpoonright J_i^1) \backslash S(i)$ is critical and $F(i) \subseteq F(J_i^1) \cup S(i)$. By Lemma 10 there exists for every $i \in K$ a set $J_i^2 \subseteq J_i^1$ such that $((F \upharpoonright J_i^2) \backslash S(i)) \backslash A'$ is critical and $F(i) \subseteq F(J_i^2) \cup S(i) \cup A'$, i.e. $(F \upharpoonright J_i^2) \backslash Y_i$ is critical and $F(i) \subseteq F(J_i^2) \cup Y_i$. With aid of Corollary 7 we get for every $i \in K$

a set J_i^3 with $J_i^2 \subseteq J_i^3 \subseteq J_\alpha$, $Y_i \subseteq F(J_i^3) \cup A'$ and $(F \upharpoonright J_i^3)\backslash A'$ critical. Consequently $F(i)\backslash A' \subseteq F(J_i^3)\backslash A'$. For $J' = \bigcup \{J_i^3 \mid i \in K\}$ we have: $(F \upharpoonright J')\backslash A'$ is critical and $F(K) \subseteq F(J') \cup A'$; and Claim 3 is proved. Now put $J = J' \cup K$.

CLAIM 4. $F \upharpoonright J$ is critical.

Clearly $IA(F \upharpoonright J) \neq \varnothing$. By Claim 3, A' is a maximal jilted subset of $F(J)$ w.r.t. $F \upharpoonright J'$. If $h \in IA(F \upharpoonright J)$, then $\operatorname{rng} h \upharpoonright L$ is a jilted subset of $F(J)$ w.r.t. $F \upharpoonright J'$. We have $|\operatorname{rng} h \upharpoonright L| = |L| = |A'|$. Further we concluded from Corollary 9 that two maximal jilted subsets have the same cardinality. Therefore $\operatorname{rng} h \upharpoonright L$ is a maximal jilted subset of $F(J)$ w.r.t. $F \upharpoonright J'$. We obtain $\operatorname{rng} h = F(J)$; this implies that $F \upharpoonright J$ is critical, and Theorem 25 is proved.

COROLLARY 26. *If $IA(F) \neq \varnothing$, then $\mathfrak{L} = (\{G \subseteq F \mid G \text{ critical}\}, \subseteq)$ is a complete atomic lattice.*

Proof. By Theorem 4.1 of [15] $\mathfrak{L}$ is a complete lattice. To prove that $\mathfrak{L}$ is atomic let G be a nonempty critical subfamily of F. By Theorem 25 we have:

$$\operatorname{dmn} G = \bigcup \{J \subseteq \operatorname{dmn} G \mid G \upharpoonright J \text{ is critical and } \exists\alpha(1 \leqslant |J\backslash J_\alpha| < \aleph_0)\}.$$

Let α be the least ordinal σ with the property that there is a set $J \subseteq \operatorname{dmn} G$ such that $G \upharpoonright J$ is critical and $1 \leqslant |J\backslash J_\sigma| < \aleph_0$. Choose $J \subseteq \operatorname{dmn} G$ with $1 \leqslant |J\backslash J_\alpha| < \aleph_0$ and $|J\backslash J_\alpha|$ minimal. If $\mathfrak{G} = \{H \subseteq G \mid \exists K \subseteq J\,(H \upharpoonright K \text{ critical})\}$, then $\bigcap \mathfrak{G}$ is a minimal element of $\{H \subseteq G \mid H \neq \varnothing \text{ and } H \text{ critical}\}$.

LEMMA 27. *Let $F = (F(i) \mid i \in I)$ be a family with $IA(F) \neq \varnothing$ and let G be a critical subfamily of F. Let $i \in \operatorname{dmn} G$ and A be a finite subset of $G(i)$ such that there exists a set $J \subseteq I$ with the property that $(F \upharpoonright J)\backslash A$ is critical and $G(i) \subseteq F(J) \cup A$. Then there already exists a set $J' \subseteq \operatorname{dmn} G$ such that $(F \upharpoonright J')\backslash A$ is critical and $G(i) \subseteq F(J') \cup A$.*

Proof. Choose $f \in IA(F)$. If $g = f \upharpoonright (J\backslash \operatorname{dmn} G)$, then, since G is critical, $\operatorname{rng} g \cap G(\operatorname{dmn} G) = \varnothing$. By assumption we have $IA((G \cap F \upharpoonright J)\backslash A) \neq \varnothing$. Let h be an element of $IA((G \cap F \upharpoonright J)\backslash A)$ and put $h^* = h \cup g$. Then $h^* \in IA((F \upharpoonright J)\backslash A)$ and $G(i)\backslash A \subseteq F(J)\backslash A = \operatorname{rng} h^*$; consequently we get $G(i)\backslash A \subseteq \operatorname{rng} h$. By Theorem 2 there exists a set $J' \subseteq \operatorname{dmn} G \cap J$ such that $(F \upharpoonright J')\backslash A'$ is critical and $G(i)\backslash A \subseteq F(J')$.

COROLLARY 28. *If F is a family with $IA(F) \neq \varnothing$ and if G is a critical subfamily of F, then* $\operatorname{rk}(i, G) = \operatorname{rk}(i, F)$ *for every $i \in \operatorname{dmn} G$.*

With aid of Corollary 28 we can prove the following corollary to Theorem 25.

COROLLARY 29. *If $F = (F(i) \mid i \in I)$ is a family, A a finite set and if $J \subseteq I$*

has the property that $(F \upharpoonright J)\backslash A$ *is critical, then* $J = \bigcup\{K \subseteq J \mid (F \upharpoonright K)\backslash A$ *critical and* $\exists\alpha(K \subseteq J(\alpha+1, F)$ *and* $1 \leqslant |K\backslash J(\alpha, F)| < \aleph_0)\}$.

Proof. By Theorem 25 it suffices to prove that $J(\alpha, F) \cap J = J(\alpha, (F \upharpoonright J)\backslash A)$. Lemma 11 yields $J(\alpha, F) = J(\alpha, F\backslash A)$ and Corollary 28 implies $J(\alpha, F\backslash A) \cap J = J(\alpha, (F \upharpoonright J)\backslash A)$. We get the desired result.

COROLLARY 30. *Let F be a family with* $\operatorname{dmn} F = I$ *and* $IA(F) \neq \varnothing$ *and let S be a finite characterization of F. If* $J \subseteq I$ *and if* $S \upharpoonright J$ *is a maximal critical subfamily of S, then* $F \upharpoonright J$ *is a maximal critical subfamily of F.*

Proof. By Theorem 5 we have $J = \bigcup\{K \Subset J \mid S \upharpoonright K \text{ critical}\}$. The proof of Theorem 25 yields for every $K \Subset J$ with $S \upharpoonright K$ critical a set $J_K \subseteq I$ with the property that $K \subseteq J_K$ and $F \upharpoonright J_K$ is critical. Put $J' = \bigcup\{J_K \mid K \Subset J$ and $S \upharpoonright K \text{ critical}\}$. Clearly $J \subseteq J'$; in addition $F \upharpoonright J'$ is critical because $IA(F) \neq \varnothing$. Consequently $S \upharpoonright J'$ is critical too. From the maximality of $S \upharpoonright J$ and $IA(F) \neq \varnothing$ we obtain $J = J'$.

THEOREM 31. *For every ordinal* α *there exists a critical family G with the following properties*:

(1) *Every critical subfamily of G different from G is the empty set.*

(2) $\operatorname{rk}(G) = \alpha + 1$.

Proof. We prove the theorem by transfinite induction on α. Put $G_0 = \{(0, \{0\})\}$. For every $\beta < \alpha$ let G_β be a nonempty minimal critical family of rank $\beta + 1$.

Case 1. α is a limit ordinal. For every $\beta < \alpha$ let G_β^* be a copy of G_β such that the copies G_β^* are pairwise disjoint. Put

$$F_\alpha = \bigcup\{G_\beta^* \mid \beta < \alpha\}.$$

Case 2. α is the successor ordinal $\beta + 1$. Let for every $n \in \omega$ the family ${G_\beta}^n$ be a copy of G_β such that the copies ${G_\beta}^n$ are pairwise disjoint, and put

$$F_\alpha = \bigcup\{{G_\beta}^n \mid n < \omega\}.$$

Now in both of the cases choose an $x \notin F_\alpha(\operatorname{dmn} F_\alpha)$ and an $i \notin \operatorname{dmn} F_\alpha$ and define G_α as follows:

$$G_\alpha(j) = \begin{cases} F_\alpha(\operatorname{dmn} F_\alpha) \cup \{x\}, & \text{if } i = j \\ F_\alpha(j) \cup \{x\}, & \text{if } j \in \operatorname{dmn} F_\alpha. \end{cases}$$

G_α has the desired properties.

6. Nash-Williams' Margin Functions

As we have already mentioned in the introduction we use our methods to prove in this paragraph a straightforward generalization of the criterion of Damerell, Milner and Nash–Williams in [2, 8, 9]. The proof shows the connection between the concept of critical family and the concept of margin function. Up to unessential modifications we follow the definitions in [9].

A family $(T_r \mid r \in R)$ is called a *directed system on a set* Y, if $Y = \bigcup \{T_r \mid r \in R\}$ and if for every two elements $r_1, r_2 \in R$ there is an element $r_3 \in R$ such that $T_{r_1} \cup T_{r_2} \subseteq T_{r_3}$.

Let $\mathfrak{P}(Y)$ denote the power set of Y and let g be a function from $\mathfrak{P}(Y)$ into $\omega \cup \{-1, \infty\}$. Further let $(T_r \mid r \in R)$ be a directed system on Y. $(T_r \mid r \in R)$ will be called *g-constant* if $g(T_{r_1}) = g(T_{r_2})$ for all $r_1, r_2 \in R$. $\mathfrak{T}(Y, g)$ will denote the set of all g-constant directed systems on Y. If $T \in \mathfrak{T}(Y, g)$, $\hat{g}(T)$ will denote the value of $g(T_r)$ for each $r \in R$. Put

$$\tilde{g}(Y) = \min\{\hat{g}(T) \mid T \in \mathfrak{T}(Y, g)\}.$$

If $F = (F(i) \mid i \in I)$ is a family, we define by transfinite induction functions m_α from $\mathfrak{P}(F(I))$ into $\omega \cup \{-1, \infty\}$ and subsets I_α of I as follows: Put

$$m_0(A) = \begin{cases} |A|, & \text{if } |A| < \aleph_0 \\ \infty, & \text{if } |A| \geqslant \aleph_0, \end{cases} \quad \text{and}$$

$$I_0 = \varnothing.$$

If α is a limit ordinal, define

$$m_\alpha(A) = \min\{m_\beta(A) \mid \beta < \alpha\} \quad \text{and}$$
$$I_\alpha = \bigcup \{I_\beta \mid \beta < \alpha\}.$$

If $\alpha = \beta + 1$, define

$$I_\alpha = \{i \in I \mid \exists A \subseteq F(I)(\tilde{m}_\beta(A) < \infty \text{ and } F(i) \subseteq A)\},$$

$$f_\beta(A) = |\{i \in I_{\beta+1} \backslash I_\beta \mid F(i) \subseteq A\}|, \quad \text{and}$$

$$m_\alpha(A) = \begin{cases} \infty, & \text{if } \tilde{m}_\beta(A) = \infty, \\ -1, & \text{if } \tilde{m}_\beta(A) < \infty \text{ and } f_\beta(A) > m_\beta(A), \\ \tilde{m}_\beta(A) - f_\beta(A) & \text{otherwise.} \end{cases}$$

If A is a set, the family (A), having A as its only member, is a directed system on A. Therefore the following holds:

(i) If $\alpha < \beta$, then $\tilde{m}_\beta(A) \leqslant \tilde{m}_\alpha(A)$.
(ii) If $\alpha < \beta$, then $I_\alpha \subseteq I_\beta$.
(iii) $\tilde{m}_\alpha(A) \leqslant m_\alpha(A)$.
(iv) If $\alpha < \beta$, then $m_\beta(A) \leqslant m_\alpha(A)$.

THEOREM 32. *If A' is a jilted subset of A* w.r.t. $F \upharpoonright I_\alpha$, *then* $m_0(A') \leqslant \tilde{m}_\alpha(A) \leqslant m_\alpha(A)$.

Proof. We prove the theorem by transfinite induction on α. If $\alpha = 0$, then $m_0(A') \leqslant m_0(A)$; $m_0(A') \leqslant \tilde{m}_0(A)$ is proved in the same way as Claim II. Now assume the inductive hypothesis that Theorem 32 becomes true if α is replaced by any smaller ordinal. Let A' be a jilted subset of A w.r.t. $F \upharpoonright I_\alpha$.

CLAIM I. $m_0(A') \leqslant m_\alpha(A)$.

Case 1. $m_\alpha(A) = \infty$. Then clearly $m_0(A') \leqslant m_\alpha(A)$.

Case 2. $m_\alpha(A) < \infty$. If α is a limit ordinal, then A' is a jilted subset of A w.r.t. $F \upharpoonright I_\beta$ for every $\beta < \alpha$ and consequently $m_0(A') \leqslant m_\beta(A)$ for every $\beta < \alpha$ by the inductive hypothesis. We obtain $m_0(A') \leqslant m_\alpha(A)$. Now let $\alpha = \beta + 1$. Then, by definition of $m_\alpha(A)$, $\tilde{m}_\beta(A) < \infty$. Choose $g \in IA((F \upharpoonright I_\alpha)\backslash A')$. $A \backslash g[I_\beta]$ is a jilted subset of A w.r.t. $F \upharpoonright I_\beta$, hence we obtain from the inductive hypothesis: $| A \backslash g[I_\beta]| = m_0(A \backslash g[I_\beta]) \leqslant \tilde{m}_\beta(A) < \infty$. If $i \in I_{\beta+1} \backslash I_\beta$ and $F(i) \subseteq A$, then $g(i) \in A \backslash (A' \cup g[I_\beta])$. It follows that $f_\beta(A) \leqslant |(A \backslash g[I_\beta]) \backslash A' | = | A \backslash g[I_\beta]| - | A' |$; consequently $f_\beta(A) \leqslant \tilde{m}_\beta(A) - | A' |$, i.e. $| A' | = m_0(A') \leqslant m_\alpha(A)$, and Claim I is proved.

CLAIM II. $m_0(A') \leqslant \tilde{m}_\alpha(A)$.

Let $(T_r \mid r \in R)$ be an m_α-constant directed system on A.

Case 1. A' is infinite. Then for every $n \in \omega$ there is an $r \in R$ such that $| A' \cap T_r | > n$. $A' \cap T_r$ is a jilted subset of T_r w.r.t. $F \upharpoonright I_\alpha$. By Claim I we have: $n < m_0(A' \cap T_r) \leqslant m_\alpha(T_r)$. We obtain $\hat{m}_\alpha((T_r \mid r \in R)) = \infty$ and therefore $\tilde{m}_\alpha(A) = \infty = m_0(A')$.

Case 2. A' is finite. Then we get for $(T_r \mid r \in R)$ an $r \in R$ with $A' \subseteq T_r$. By Claim I we obtain $m_0(A') \leqslant m_\alpha(T_r)$ and consequently $\hat{m}_\alpha((T_r \mid r \in R)) \geqslant m_0(A')$. This implies $m_0(A') \leqslant \tilde{m}_\alpha(A)$; and Theorem 32 is proved.

COROLLARY 33. *If* $IA(F) \neq \varnothing$, *then* $m_\alpha(A) \geqslant 0$ *for every ordinal* α *and every* $A \subseteq F(\operatorname{dmn} F)$.

Corollary 33 says that the criterion of Damerell, Milner and Nash–Williams is necessary. We will now prove that it is sufficient.

LEMMA 34. *If* $IA(F \upharpoonright J_\alpha) \neq \varnothing$, *then the following holds*:

(1) *If* $J \subseteq J_\alpha$ *and if* A' *is a jilted subset of* $F(I)$ w.r.t. $F \upharpoonright J_\alpha$ *such that* $(F \upharpoonright J)\backslash A'$ *is critical, then* $\tilde{m}_\alpha(F(J) \cup A') \leqslant m_0(A')$.

(2) $J_{\alpha+1} = I_{\alpha+1}$.

Proof. For $\alpha = 0$ (1) is trivial and (2) is easy to prove. We assume the inductive hypothesis

Lemma 34 becomes true if α is replaced by any ordinal β with $\beta < \alpha$. (*)
First we prove with aid of (*):

CLAIM A. Let A' be a jilted subset of $F(I)$ w.r.t. $F \upharpoonright J_\alpha$ and let $K \subseteq J_\alpha$ be a set such that $(F \upharpoonright K)\backslash A'$ is critical, $K \subseteq J_{\beta+1}$ and $1 \leqslant |K\backslash J_\beta| < \aleph_0$ for some $\beta < \alpha$. Then $m_\alpha(F(K) \cup A') \leqslant m_0(A')$.

Claim A is obviously true for $\alpha = 0$. Suppose that $\alpha > 0$. If $m_0(A') = \infty$, then clearly $m_\alpha(F(K) \cup A') \leqslant m_0(A')$. Now assume $m_0(A') = |A'| < \infty$. Choose $g \in IA((F \upharpoonright J_\alpha)\backslash A')$ and define $A'' = A' \cup g[K\backslash J_\beta]$. Then we have $m_0(A'') = |A''| < \infty$. Since $(F \upharpoonright K)\backslash A'$ is critical, $(F \upharpoonright K \cap J_\beta)\backslash A''$ is critical too. Note that $F(K) \cup A' = F(K \cap J_\beta) \cup A''$ and that A'' is a jilted subset of $F(I)$ w.r.t. $F \upharpoonright J_\beta$. Therefore we obtain from (*):

$$\tilde{m}_\beta(F(K) \cup A') = \tilde{m}_\beta(F(K \cap J_\beta) \cup A'') \leqslant |A''| < \infty \quad \text{and} \quad I_{\beta+1} = J_{\beta+1}.$$

If $i \in K\backslash J_\beta$, then $F(i) \subseteq F(K) \cup A'$. Hence $f_\beta(F(K) \cup A') \geqslant |K\backslash J_\beta| = |A''\backslash A'|$ and consequently $m_{\beta+1}(F(K) \cup A') = \tilde{m}_\beta(F(K) \cup A') - f_\beta(F(K) \cup A') \leqslant |A''| - |A''\backslash A'| = |A'|$. This yields $m_\alpha(F(K) \cup A') \leqslant |A'|$, since $\beta < \alpha$.
Now we prove (1).

If $m_0(A') = \infty$, there is nothing to prove. Assume that $m_0(A') = |A'| < \infty$. Define

$$R = \{K \subseteq J \mid (F \upharpoonright K)\backslash A' \text{ critical and } \exists \beta < \alpha (K \subseteq J_{\beta+1} \text{ and } 1 \leqslant |K\backslash J_\beta| < \aleph_0)\}.$$

Since, by assumption, $F(J)\backslash A'$ is critical, we obtain from Corollary 29 that $(F(K) \cup A' \mid K \in R)$ is a directed system on $F(J) \cup A'$. Claim *A* yields $m_\alpha(F(K) \cup A') \leqslant m_0(A') \leqslant |A'|$ for every $K \in R$. If $K \in R$, then there is a $\beta < \alpha$ with $K \subseteq J_{\beta+1} = I_{\beta+1} \subseteq I_\alpha = J_\alpha$. Hence A' is a jilted subset of $F(K) \cup A'$ w.r.t. $F \upharpoonright I_\alpha$. By Theorem 32 we obtain $|A'| = m_0(A') \leqslant m_\alpha(F(K) \cup A')$ and consequently $m_\alpha(F(K) \cup A') = |A'|$ for every $K \in R$. This implies

$$\hat{m}_\alpha((F(K) \cup A' \mid K \in R)) = |A'|.$$

Thus $\tilde{m}_\alpha(F(J) \cup A') \leqslant |A'|$, which completes the proof of (1).
Next we prove (2).

CLAIM B. $I_{\alpha+1} \subseteq J_{\alpha+1}$.

Let $i \in I_{\alpha+1}$. Then there is a set A such that $\tilde{m}_\alpha(A) < \infty$ and $F(i) \subseteq A$. Since $I_\alpha = J_\alpha$, we obtain by Theorem 32

$$|B| = m_0(B) \leqslant \tilde{m}_\alpha(A) < \infty$$

for every jilted subset B of A w.r.t. $F \upharpoonright J_\alpha$. Hence there is a maximal jilted subset A' of A w.r.t. $F \upharpoonright J_\alpha$ since $IA(F \upharpoonright J_\alpha) \neq \varnothing$. By Theorem 2 there is a set $J \subseteq J_\alpha$ such that $(F \upharpoonright J)\backslash A'$ is critical and $A \subseteq F(J) \cup A'$. This implies $F(i) \subseteq F(J) \cup A'$ and we obtain $i \in J_{\alpha+1}$.

CLAIM C. $J_{\alpha+1} \subseteq I_{\alpha+1}$.

Let $i \in J_{\alpha+1}$. Then there are a finite set A and a set $J \subseteq J_\alpha$ such that $(F \upharpoonright J)\backslash A$ is critical and $F(i) \subseteq F(J) \cup A$. By Lemma 8 we have $| F(i)\backslash \operatorname{rng} f | \leqslant | A |$ for every $f \in IA(F \upharpoonright J_\alpha)$. Hence there is finite maximal jilted subset A' of $F(i)$ w.r.t. $F \upharpoonright J_\alpha$. Theorem 2 yields a set $J \subseteq J_\alpha$ such that $(F \upharpoonright J)\backslash A'$ is critical and $F(i) \subseteq F(J) \cup A'$. From (1) we obtain $\tilde{m}_\alpha(F(J) \cup A') \leqslant m_0(A') = | A' | < \infty$, consequently $i \in I_{\alpha+1}$; and Lemma 34 is proved.

THEOREM 35. *If F is a family with* $\operatorname{dmn} F = I$ *and* $| I\backslash I_H | \leqslant \aleph_0$, *then F has an injective choice function iff $m_\alpha(A) \geqslant 0$ for every $A \subseteq F(I)$ and every ordinal α.*

Proof. If $IA(F) \neq \varnothing$, then, by Corollary 33, we have $m_\alpha(A) \geqslant 0$ for every $A \subseteq F(I)$ and for every α.

To prove the converse assume that F is a family with $\operatorname{dmn} F = I$ and $m_\alpha(A) \geqslant 0$ for every $A \subseteq F(I)$ and every α. By Theorem 22 it suffices to prove $IA(F \upharpoonright I_H) \neq \varnothing$, and Theorem 20 shows that we are done if we prove for every α the following

CLAIM. If $IA(F \upharpoonright J_\alpha) \neq \varnothing$, then $IA(F \upharpoonright J_\alpha \cup K) \neq \varnothing$ for every $K \Subset J_{\alpha+1}$.

Assume the contrary. Then there are L and i such that $L \Subset J_{\alpha+1}\backslash J_\alpha$, $IA(F \upharpoonright J_\alpha \cup L) \neq \varnothing$, $i \in J_{\alpha+1}\backslash J_\alpha$ and $IA(F \upharpoonright J_\alpha \cup L \cup \{i\}) = \varnothing$. By Theorem 2 there is a set $J \subseteq J_\alpha \cup L$ with $F \upharpoonright J$ critical and $F(i) \subseteq F(J)$. Choose $g \in IA(F \upharpoonright J_\alpha \cup L)$. Then $g[J\backslash J_\alpha]$ is a maximal jilted subset of $F(J)$ w.r.t. $F \upharpoonright J_\alpha$. From Theorem 32 and Lemma 34 we obtain $\tilde{m}_\alpha(F(J)) = | g[J\backslash J_\alpha]|$. Further we have $(J\backslash J_\alpha) \cup \{i\} \subseteq \{k \in I \mid F(k) \subseteq F(J)$ and $k \in J_{\alpha+1}\backslash J_\alpha\}$. Consequently $\tilde{m}_\alpha(F(J)) = | g[J\backslash J_\alpha]| < |(J\backslash J_\alpha) \cup \{i\}| \leqslant f_\alpha(F(J))$ and therefore $m_{\alpha+1}(F(J)) = -1$; this is a contradiction, and Theorem 35 is proved.

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